ELLIPTIC THEORY ON MANIFOLDS WITH CORNERS: I. DUAL MANIFOLDS AND PSEUDODIFFERENTIAL OPERATORS

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Abstract. In this first part of the paper, we define a natural dual object for manifolds with corners and show how pseudodifferential calculus on such manifolds can be constructed in terms of the localization principle in C^*-algebras. In the second part, these results will be applied to the solution of Gelfand's problem on the homotopy classification of elliptic operators for the case of manifolds with corners.

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Introduction

This paper deals with elliptic theory on manifolds with corners.

Such manifolds arise, e.g., if one supplements the class of closed manifolds by manifolds with boundary and considers products of manifolds. A natural class of operators on such manifolds was introduced by Melrose [7, 8]. Operators on manifolds with corners have been actively studied, e.g. see [1, 3, 4, 5, 6, 9, 10, 11, 12, 14].

The present paper consists of two parts. In the first part, we define a natural dual object for manifolds with corners and show how pseudodifferential calculus on such manifolds can be constructed in terms of the localization principle in C^*-algebras. In the second part, these results will be applied to the solution of Gelfand's problem

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on the homotopy classification of elliptic operators for the case of manifolds with corners.

In more detail, the outline of the first part is as follows. In Sec. 1, we deal with the geometry of manifolds with corners. Specifically,

- In Sec. 1.1 we recall some facts and definitions concerning manifolds with corners. Most of the material in this section is not new, except possibly in form.
- In Sec. 1.2 we introduce a new geometric object, the dual manifold $M^{\#}$ of a manifold M with corners, and study some structures on it. The importance of this space lies in the fact that, on the one hand, pseudodifferential operators on manifolds with corners can be naturally defined as operators local with respect to the action of the algebra of continuous functions on the dual manifold. On the other hand, as will be shown in the second part of paper, under an additional assumption the K-homology of the dual manifold $M^{\#}$ classifies the elliptic theory on M.

In Sec. 2 we define zero-order pseudodifferential operators (ψDO) in L^2 spaces on manifolds with corners. The definition is based on the localization principle in C^*-algebras (e.g., see [15, Proposition 3.1]), goes by induction over the depth of the manifold, i.e., the maximum codimension of the strata (one starts from smooth manifolds, which have depth zero), and naturally involves parameter-dependent ψDO (which serve as symbols for ψDO at subsequent inductive steps). Hence we need some preliminaries:

- In Sec. 2.1 we introduce L^2 spaces on manifolds with corners.
- In Sec. 2.2 we discuss translation-invariant operators in vector bundles over manifolds with corners and their relationship with parameter-dependent operators.
- In Sec. 2.3 we present the adaptation [13] of the localization principle to operator families. The proofs are either contained in [13] or can be obtained from those in [13] by obvious modifications; hence we omit them altogether.

After that, in Sec. 2.4 we give the definition of ψDO and prove their properties.

Nomenclature. We shall use the following notation.

$L^2(X, \mu, H)$ is the space of square integrable H-valued functions on a metric space X with respect to a measure μ (where H is a Hilbert space). We omit the argument H if $H = \mathbb{C}$ and also omit μ if it is clear from the context.

$\mathcal{B}H$ and $\mathcal{K}H$ are the algebra of bounded operators and the ideal of compact operators in a Hilbert space H.

$C(X, \mathcal{A})$ is the C^*-algebra of continuous bounded functions on X ranging in the C^*-algebra $\mathcal{A}$, and $C_0(X, \mathcal{A})$ is the subalgebra of functions decaying at infinity. We omit the argument $\mathcal{A}$ if $\mathcal{A} = \mathbb{C}$.

1. Geometry

1.1. Manifolds with corners and their faces

Definition 1.1. A *manifold of dimension n with corners* is a Hausdorff topological space M in which each point x has a coordinate neighborhood of the form $\overline{\mathbb{R}}^d_+ \times \mathbb{R}^{n-d}$, $d = d(x) \in \{0, \dots, n\}$ where x is represented by the origin. Moreover, the transition maps are smooth functions. Unless specified otherwise, we assume that M is connected and compact. The maximum number d is called the *depth* of the manifold and will be denoted by $k = k(M)$.

Some examples of manifolds with corners are shown in Fig. 1.1.

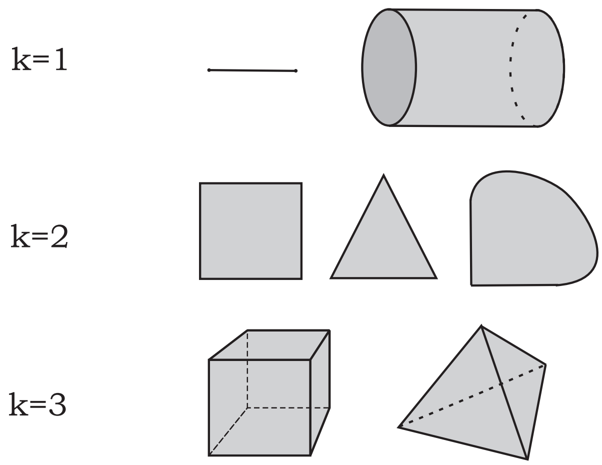

FIGURE 1.1. Manifolds with corners of depth k.

Local defining functions. By definition, each point $x \in M_j$ has a neighborhood $U \subset M$ with local coordinates $\rho_1, \ldots, \rho_n$ such that the manifold is determined in these coordinates by the system of inequalities

$$\rho_1 \geq 0, \ldots, \rho_j \geq 0. \tag{1.1}$$

The coordinates $(\rho_1, \ldots, \rho_j)$ are called *defining functions* of the face $\overline{\mathbb{R}}^j_+ \subset \overline{\mathbb{R}}^n_+$.

Faces. The set

$$\{x \in M \colon d(x) = l\}$$

is a smooth manifold of codimension l in M. Its connected components are called *open faces of codimension* l. Let $\Gamma^\circ_j(M)$, $j = 0, \ldots, N$ be all possible open faces of M, and let d_j be their codimensions. We assume that $d_0 = 0$ (so that $\Gamma^\circ_0(M) = M^\circ$ is the interior of M) and $d_j > 0$ for $j > 0$. Thus M is represented as the disjoint union

$$M = \bigsqcup_{j=0}^{N} \Gamma^\circ_j(M) \equiv M^\circ \sqcup \partial M,$$

$$\text{where } \partial M = \bigsqcup_{j=1}^{N} \Gamma^\circ_j(M) \text{ is the } \textit{boundary} \text{ of } M.$$

Faces of codimension one are called *hyperfaces*.

Proposition 1.2. *There exist canonically defined manifolds $\Gamma_j(M)$ with corners such that $\Gamma^\circ_j(M)$ is the interior of $\Gamma_j(M)$ and the diagram*

$$\begin{tikzcd}
\Gamma^\circ_j(M) \arrow[r] \arrow[dr, "i^\circ_j"'] & \Gamma_j(M) \arrow[d, dashed, "i_j"] \\
& M,
\end{tikzcd}$$

where the horizontal arrow and i°_j are natural embeddings and i_j is an immersion of manifolds with corner, commutes. The manifold $\Gamma_j(M)$ is called a closed face of M.

Proof is given in the Appendix. $\square$

Since $\Gamma_j(M)$ is a compact manifold with corners, we have

$$\partial\Gamma_j(M) = \bigsqcup_{l=1}^{L} \Gamma_l^\circ(\Gamma_j(M)).$$

The image under i_j of each open face $\Gamma_l^\circ(\Gamma_j(M))$, $l > 0$, of th manifold $\Gamma_j(M)$ coincides with some open face $\Gamma_r^\circ(M)$, $r = r(l)$, of M with $d_r > d_j$. in this case, we say that the face Γ_j (or Γ_j°) and Γ_r (or Γ_r°) are *adjacent to each other* and write $\Gamma_j \succ \Gamma_r$. The restriction

$$i_{jl} := i_j|_{\Gamma_l^\circ(\Gamma_j(M))} \colon \Gamma_l^\circ(\Gamma_j(M)) \longrightarrow \Gamma_{r(l)}^\circ(M)$$

is a finite covering whose structure group is a quotient of the homotopy group $\pi_1(\Gamma_{r(l)}^\circ(M))$ and a subgroup of the permutation group $\mathfrak{S}_m$, where m is the number of sheets of the cover.

The compressed cotangent bundle. The compressed cotangent bundle T^*M of a manifold M with corners is defined in the usual way (see [7]). We take the subspace $\mathrm{Vect}_b(M)$ of the space $\mathrm{Vect}(M)$ of vector fields on M formed by vector fields tangent to all open faces. The subspace $\mathrm{Vect}_b(M)$ is a locally free $C^\infty(M)$-module.

Indeed, in local coordinates

$$(\rho_1, \dots, \rho_d, y_{d+1}, \dots, y_n) \in \overline{\mathbb{R}}_+^d \times \mathbb{R}^{n-d}$$

a local basis in $\mathrm{Vect}_b(M)$ is formed by the vector fields

$$\rho_1 \frac{\partial}{\partial \rho_1}, \dots, \rho_d \frac{\partial}{\partial \rho_d}, \frac{\partial}{\partial y_{d+1}}, \dots, \frac{\partial}{\partial y_n}.$$

Consequently, $\mathrm{Vect}_b(M)$ is the section space of some vector bundle on M, which will be denoted by TM (the *extended cotangent bundle* of M), and the *compressed cotangent bundle* T^*M is now defined as the bundle ($\mathbb{R}$-) dual to TM. In the local coordinates $(\rho_1, \dots, \rho_d, y_{d+1}, \dots, y_n)$, a basis in the module of sections of T^*M is given by the forms

$$\rho_1^{-1} d\rho_1, \dots, \rho_d^{-1} d\rho_d, dy_{d+1}, \dots, dy_n.$$

Conormal bundles of faces. Let $F = \Gamma_j^\circ(M)$ be an open face of codimension $d = d_j$ in M. We define the *conormal bundle* of F as the subset $N^*F \subset T^*M|_F$ formed by functionals ξ vanishing on any vector $v \in TM|_F$ that can be continued to a vector field second-order tangent to all faces in ∂M. One readily sees that N^*F is indeed a vector bundle; a basis in its fiber consists of the 1-forms

$$\rho_1^{-1} d\rho_1, \dots, \rho_d^{-1} d\rho_d.$$

This bundle can be canonically extended to a bundle over the closed face $\overline{F}$; the latter bundle is called the *conormal bundle* of $\overline{F}$ and is denoted by $N^*\overline{F}$.

Proposition 1.3. *One has the canonical direct sum decomposition*

$$T^*M|_{\overline{F}} = T^*\overline{F} \oplus N^*\overline{F}$$

(*where the bundle on the left-hand side is obtained as the pullback under the immersion of $\overline{F}$ in M*).

Proof. The assertion is local, so that we can assume that $\overline{F}$ is embedded in M. Then the embedding $T^*\overline{F} \subset T^*M$ is obtained as the map dual to the restriction

$$\mathrm{Vect}_b(M) \longrightarrow \mathrm{Vect}_b(\overline{F})$$

of vector fields in $\mathrm{Vect}_b(M)$to $\overline{F}$. Now the desired properties can be verified in coordinates. $\square$

Normal bundles of faces. Let $F = \Gamma_j^\circ(M)$ be again an open face of codimension $d = d_j$ in M, and let (ρ, y) and $(\widetilde{\rho}, \widetilde{y})$ be two coordinate systems on M in a neighborhood of some point in F. Since $\rho = \widetilde{\rho} = 0$ on F, we see that the change of variables $(\rho, y) \longmapsto (\widetilde{\rho}, \widetilde{y})$ has the form

$$\widetilde{y} = f(y) + O(\rho), \quad \widetilde{\rho} = A(y)\rho + O(\rho^2), \tag{1.2}$$

where $A(y)$ is a smooth $d \times d$ matrix function. The mapping (1.2) should take the positive quadrant with respect to the variable ρ to itself, and hence, letting ρ tend to zero, we verify that

$$A(y) = \Pi(y)\Lambda(y),$$

where $\Pi(y)$ is a permutation matrix (and hence is locally constant in y) and

$$\Lambda(y) = \operatorname{diag}\{\lambda_1(y), \dots, \lambda_d(y)\}$$

is a diagonal matrix with positive entries. The cocycle condition for the matrices $A(y)$ implies that the matrices $\Pi(y)$ themselves satisfy the same cocycle condition, so that we can define the d-dimensional real vector bundle NF over F for which the matrices $\Pi(y)$ are the transition mappings. The change of variables

$$t_j = -\ln \rho_j, \quad j = 1, \dots, d,$$

clarifies the meaning of this bundle. The second component in (1.2) becomes

$$\widetilde{t} = \Pi(y)t + \ln \Lambda(y) + O(e^{-2t}) = \Pi(y)t + O(1), \qquad t_j \to +\infty, \quad j = 1, \dots, d.$$

Thus NF is the "bundle of logarithms of determining functions" of the submanifold F. We call it the *logarithmic normal bundle* of F. The matrices Π simultaneously specify a bundle of positive quadrants $\overline{\mathbb{R}}^d_+$ over F, which we denote by N_+F and call the *normal bundle* of F. We have the exponential mapping

$$\begin{aligned} \exp : NF &\longrightarrow N_+F, \\ (y, t) &\longmapsto (y, \exp(-t)) = (y, e^{-t_1}, \dots, e^{-t_d}), \end{aligned}$$

which diffeomorphically maps the first bundle onto the interior of the second.

One readily sees that both bundles naturally extend to bundles $N\overline{F}$ and $N_+\overline{F}$ over the closed face $\overline{F}$.

By construction, the structure group of these bundles is a subgroup $\mathfrak{S}_{\overline{F}}$ of the permutation group $\mathfrak{S}_d$. (Thus the numbering of the coordinates ρ in all charts is chosen in such a way that the transition matrices range in the subgroup $\mathfrak{S}_{\overline{F}}$).

Remark. We shall assume that the bundles $N\overline{F}$ and $N^*\overline{F}$ are reduced to the minimal possible permutation structure group $\mathfrak{S}_{\overline{F}}$. This will be used in the sequel (in particular, see Lemma A.2).

Proposition 1.4. *The logarithmic normal bundle $N\overline{F}$ and the conormal bundle $N^*\overline{F}$ are canonically dual.*

Proof will be given in the Appendix. □

Remark. The bundles $N\overline{F}$ and $N^*\overline{F}$ viewed as bundles with the structure group $\mathfrak{S}_{\overline{F}}$ are canonically isomorphic, since permutation matrices are unitary.

Compatible exponential mappings. For each closed face $\Gamma_j(M)$ of a manifold M with corners, we have defined the normal bundle $N_+\Gamma_j(M)$. Just as with submanifolds of smooth manifolds, one can define *exponential mappings* of these bundles into the manifold M itself, which are local diffeomorphisms in a neighborhood of the zero section. Moreover, for adjacent faces these diffeomorphisms will be compatible in some sense. More precisely, the following theorem holds.

Theorem 1.5. *Let $\varepsilon > 0$ b sufficiently small. Then there exist smooth mappings*

$$f_j \colon N_+\Gamma_j(M) \longrightarrow M, \quad j = 1, \dots, N,$$

defined for $|\rho| \le \varepsilon$, where ρ is the variable in the fiber of the bundle $N_+\Gamma_j(M)$, such that the following conditions hold:

1. *On the zero section, $f_j = i_j$.*
2. *f_j is a local diffeomorphism.*
3. *The restriction $f_j|_U$ of the mapping f_j to some neighborhood of the open face $\Gamma_j^\circ(M)$ in $N_+\Gamma_j(M)$ is a diffeomorphism.*
4. *If $\Gamma_j(M) \succ \Gamma_l(M)$, then the mappings f_j and f_l are locally compatible in the following sense. In a neighborhood of any point $x \in \Gamma_l(M)$, the diagram*

$$\begin{tikzcd} N_+\Gamma_j(M) \arrow[r, "f_l^{-1}\circ f_j"] \arrow[d, "\pi_1"'] & N_+\Gamma_l(M) \arrow[d, "\pi_2"] \\ \Gamma_j(M) \arrow[r, "f_l^{-1}\circ f_j"'] & \varphi(\Gamma_j(M)) \end{tikzcd} \tag{1.3}$$

commutes, where π_1 is the natural projection and π_2 is the projection in the fibers of $N_+\Gamma_l(M)$ onto the coordinate subbundle into which $\Gamma_j(M)$ is mapped under the local diffeomorphism $\varphi = f_l^{-1} \circ f_j$, along the complementary coordinate subbundle.

Proof will be given in the Appendix. □

Remark. (a) Let $\Gamma_j(M) \succ \Gamma_r(M)$. Since

$$N_+\Gamma_l^\circ(\Gamma_j(M)) \subset i_{jl}^* N_+\Gamma_r(M), \quad r = r(l),$$

we see that by specifying a compatible tuple of exponential mappings f_j for the strata of M we automatically specify such tuples for the strata of any closed stratum of M.

(b) the composition

$$\widetilde{f_j} = f_j \circ \{t \mapsto e^{-t}\} \colon N\Gamma_j(M) \longrightarrow M$$

will also be referred to as the exponential map.

Corollary 1.6. *The manifold M can be covered by finitely many coordinate neighborhoods U with coordinates $\rho_U = (\rho_1, \dots, \rho_n)$ such that M is given in these coordinates by the system of inequalities* (1.1) *and the following compatibility condition holds. Suppose that two charts U and U' have a nonempty intersection.*

(1) *If the number of defining functions in U and U' is the same, then they coincide in $U \cap U'$ up to a permutation;*
(2) *otherwise, the smaller set of defining functions is a subset of the larger set in $U \cap U'$.*

Remark 1.7. This assertion plays in the theory of manifolds with corners the same role as the collar neighborhood theorem in the theory of manifolds with boundary (and contains the latter for the case in which the depth k is equal to one.

To be definite, we assume in the following that the defining functions specify coordinates in the domain where they are less than $3/2$.

1.2. **The dual manifold $M^\#$ and the algebra $C(M^\#)$**

Definitions. On the space $\mathbb{R}^k_t \times \mathbb{R}^m_x$, we define the algebra $C(k,m)$ of bounded continuous functions $f(t,x)$ such that

$$f(\omega|t|, x) \longrightarrow F(\omega) \quad \text{as } |t| \to \infty$$

uniformly with respect to x and $\omega = t/|t|$, where $F(\omega)$ is some (continuous) function.

In the algebra of continuous functions on the interior M° of the manifold M, we single out a subalgebra $C(M^\#)$ as follows. We say that $f \in C(M^\#)$ if for each coordinate neighborhood $U \simeq \mathbb{R}^k_+ \times \mathbb{R}^{n-k}$ on M the function

$$F(t,x) = f|_U(e^{-t_1}, \dots, e^{-t_k}, x_{k+1}, \dots, x_n)$$

can be extended to a function in $C(k, n-k)$.

One can readily see that each function $f \in C(M^\#)$ is constant on each hyperface of M.

One can readily describe the space $M^\#$ of maximal ideals of the algebra $C(M^\#)$. As a set, it is the disjoint union of the interior M° of the manifold M and the following sets $F^\#$ corresponding to faces F of positive codimension.

- To each hyperface F, there corresponds a singleton $F^\#$.
- To each face F of codimension $k = \operatorname{codim} F > 1$, there corresponds a set $F^\#$ that is the quotient of the open $k-1$-simplex

$$\overset{\circ}{\triangle}_{k-1} = \Big\{ x \in \mathbb{R}^k \colon x_i > 0,\, i = 1, \dots, k, \quad \sum x_i = 1 \Big\}$$

  by the action of the structure group $\mathfrak{S}_F$ of the bundle NF.

The topology on $M^\#$ can be defined as follows. A sequence $z_n \in M^\circ$ converges to a point $z \in F^\#$ if for each $\varepsilon > 0$ all points z_n lie in the image of the neighborhood $\{|\rho| < \varepsilon\} \subset N_+F$ starting from some moment and

$$\operatorname{dist}(\ln r_1(z_n) : \dots : \ln r_k(z_n), z) \to 0.$$

(In the last formula, the line, treated as a point of the projective space, is identified with the point of the simplex through which it passes, and z is understood as a $\mathfrak{S}_F$-orbit, $z \subset \overset{\circ}{\triangle}_{k-1}$.) Finally, the adjacency conditions for the sets $F^\#$ naturally follow from the adjacency of the corresponding faces and are induced by embeddings of simplices of various dimensions.

Example 1.8. (1) If M is a manifold with boundary, then $C(M^\#)$ is the algebra of functions constant on connected components of the boundary. Hence $M^\#$ is obtained by retracting each boundary component into a point.

(2) Duality relates the cube to the octahedron and the icosahedron to the dodecahedron. The tetrahedron proves to be self-dual.

Fibered structure on the dual space. Here we assume that M is a manifold with corners such that the normal bundle of each face is trivial and show that a neighborhood of each simplex $F^\#$ of the dual manifold $M^\#$ is fibered over $F^\#$ with fiber being a cone. This result will be used only in the proof of the classification theorem in the second part of this paper.

Let $F \subset M$ be an open face of codimension j. We shall construct a neighborhood $U^\#$ of the simplex $F^\#$ in the dual manifold $M^\#$.

First, we construct a neighborhood $U \subset N_+F$. It is convenient to use the logarithmic coordinates

$$
\begin{array}{rcl}
\ln : N_+F \setminus F & \xrightarrow{\simeq} & NF, \\
(x, \rho_F) & \mapsto & (x, y = -\ln \rho_F).
\end{array}
\tag{1.4}
$$

Here $\rho_F = (\rho_1, \dots, \rho_j)$ is the set of defining functions of F. By virtue of the triviality assumption, it is globally defined.

The image of the set in which $\rho_l < 1$ for all $1 \le l \le j$ will be denoted by $N'_+F \subset NF$. In the coordinates y, it is given by the condition $y > 0$.

We use similar coordinates in neighborhoods of all faces of the face $\overline{F}$. Then in the space N'_+F we obtain the following coordinates: the coordinates $y \in \mathbb{R}^j_+$ in the fibers; the coordinates in the neighborhood $\mathbb{R}^l_+ \times \mathbb{R}^{n-j-l} \subset F$ of codimension l in F, which will be denoted by

$$
(x, \omega), \text{ where } (x_1, \dots, x_l) = -\ln(\rho_1, \dots, \rho_l).
$$

(The coordinates x are uniquely determined up to permutation; the number of these coordinates is determined by the codimension in F of the face near which the point sits.)

To construct the neighborhood U, on F we define the function $|x| := \sum_s x_s$. This is invariant under permutations of defining functions and hence well defined.

Now we globally define a set $U \subset N_+F$ by the condition

$$
U = \Big\{ (y, x, \omega) \in N'_+F \mid \min y > |x| + 1 \Big\}
$$

in local coordinates, where $\min y$ is the minimum of the coordinates $y_1, \dots, y_j$. By way of example, Fig. 1.2 shows the case in which the manifold with corners is a 1-gon; the set U corresponding to the one-dimensional edge is shown in the lower part of the figure as a dashed infinite domain.

Consider the space

$$
M^{\#}_{\ge j} = M^{\#} \setminus \bigcup_{j'=1}^{j-1} M^{\#}_{j'},
$$

obtained from $M^{\#}$ by deleting all simplices of dimension $\le j-2$.

Lemma 1.9. (1) *the restriction of the projection $p : N_+F \to M$ to U is one-to-one* (*i.e., U can also be treated as an open set in M; see top left in Fig.* 1.2)*;*

(2) *The dual space $U^{\#} \subset M^{\#}$ is an open neighborhood of the open simplex $F^{\#}$ in $M^{\#}_{\ge j}$* (*see top right in Fig.* 1.2)*.*

Proof. Let us prove that p is one-to-one. This can be violated only where distinct parts of F meet each other. We should prove that the projections of components of U corresponding to two adjacent faces are disjoint. Indeed, let U be defined in the first part by the condition

$$
\min y > |x| + 1.
$$

Then in the second part some of the coordinates x_I are interchanged with some of the coordinates y_I for some nonempty index set I. Then the set U in the second part is described by the inequality

$$
\min(x_I, y_{\overline{I}}) > |y_I| + |x_{\overline{I}}| + 1
$$

(in the original coordinates). Writing out these two systems componentwise, we see that they are inconsistent, so that the projections of the parts of U into M are disjoint.

The second assertion holds by construction. $\square$

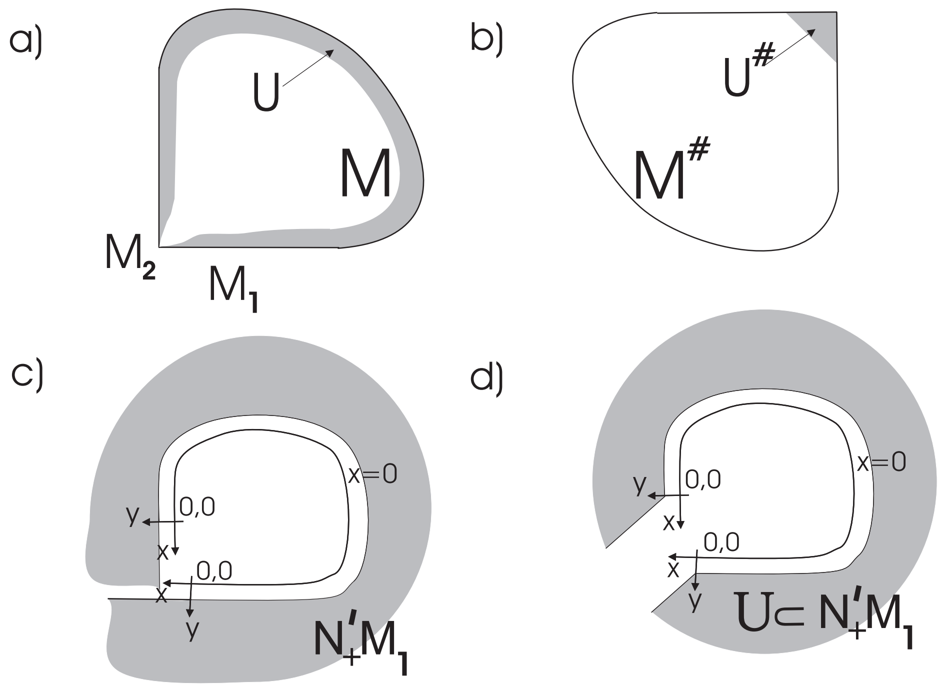

FIGURE 1.2. a) the manifold M; b) the dual space $M^{\#}$; c) the positive quadrant in the normal bundle N'_+M_1; d) the neighborhood $U \subset N'_+M_1$.

Now we can prove that the neighborhood $U^{\#}$ of the stratum $F^{\#}$ is homeomorphic to the product of $F^{\#}$ by the cone

$$K_\Omega = [0,1) \times \Omega/\{0\} \times \Omega.$$

Here the base Ω of the cone is the dual space $\overline{F}^{\#}$ of the closed face $\overline{F}$. The dual manifold is well defined, since the closed face is a manifold with corners. As a result, we find that $M^{\#}$ is a stratified manifold with singularities.

Proposition 1.10. *The projection*

$$\widetilde{p} : U^{\#} \to F^{\#}, \quad \widetilde{p}(y) := y/|y|,$$

is well defined. Its fiber is the cone $K_{\overline{F}^{\#}}$, and there is a homeomorphism

$$U^{\#} \simeq F^{\#} \times K_{\overline{F}^{\#}}.$$

Proof. A straightforward computation shows that the projection $\widetilde{p}$ is well defined. We define the map

$$\begin{array}{ccc} U & \longrightarrow & F^{\#} \times (0,1) \times F, \\ (y,x,\omega) & \mapsto & \left[\dfrac{y}{|y|}, \dfrac{|x|+1}{\min y}, (x,\omega)\right], \end{array}$$

The inverse map has the form

$$\begin{array}{ccc} F^{\#} \times (0,1) \times F & \longrightarrow & U, \\ (\theta,r,x,\omega) & \mapsto & \left[\dfrac{\theta}{\min\theta} \dfrac{|x|+1}{r}, x, \omega\right], \end{array}$$

A routine verification of the fact that these mappings extend to homeomorphisms $U^{\#} \simeq F^{\#} \times K_{\overline{F}^{\#}}$ is left to the reader. $\square$

2. Pseudodifferential Operators

2.1. **The space $L^2(M)$.** Our ψDO will act in the space $L^2(M)$, which is defined as follows.

Let M be a compact manifold with corners, and let $d\mathrm{vol}$ be a smooth measure on M (obtained, say, via an embedding of M in a compact Riemannian manifold). Now for each point $x \in M$ we define a measure μ_x in $M^\circ \cap U_x$, where $U_x \simeq V \subset \overline{\mathbb{R}}^k_+ \times \mathbb{R}^{n-k}$, $k = d(x)$, is a coordinate neighborhood of x, by setting

$$\mu_x = (\rho_1 \rho_2 \cdots \rho_k)^{-1} d\mathrm{vol}, \tag{2.1}$$

where $\rho_1, \dots, \rho_k$ are the coordinates in the $\mathbb{R}_+$-factors. Next, we take a finite cover $M = \bigcup_{j=1}^{N'} U_{x_j}$ and a subordinate partition of unity $\{e_j\}$ and set

$$\mu = \sum_{j=1}^{N'} e_j \mu_{x_j}. \tag{2.2}$$

This measure is up to equivalence independent of the ambiguity in the construction, and hence the space $L^2(M) \stackrel{\text{def}}{\equiv} L^2(M^\circ, \mu)$ is well defined up to norm equivalence. For the following, we choose and fix such a measure and hence a Hilbert space structure in $L^2(M)$. Note that the interiors of M and $M^\#$ are the same, and so $L^2(M)$ can also be viewed as $L^2(M^\#)$ (with respect to the same measure). Hence it bears the natural structure of a $C(M^\#)$-module.

2.2. **Translation-invariant operators.** In this section we shell work in the category whose objects are arithmetic spaces $\mathbb{R}^s$ and whose morphisms are linear mappings taking each standard basis vector to zero or to some standard basis vector. In particular, the automorphism group of $\mathbb{R}^s$ in this category is exactly the permutation group $\mathfrak{S}_s$ on the s standard basis vectors.

Let M be a connected compact manifold with corners, and let $E \longrightarrow M$ be a bundle with fiber $\mathbb{R}^s$ on M. We reduce E to a minimal structure group $\mathcal{G} \subset \mathfrak{S}_s$ (which is uniquely determined up to conjugacy) and consider the principal $\mathcal{G}$-bundle $\pi : \widetilde{M} \longrightarrow M$ associated with E. The following assertion is routine.

Proposition 2.1. *The space $\widetilde{M}$ is a connected manifold with corners equipped with the natural action of $\mathcal{G}$ given in any chart $U \times \mathcal{G}$ on $\widetilde{M}$, where U is a chart on M, by the formula $\sigma(z, g) = (z, g\sigma^{-1})$, $\sigma \in \mathcal{G}$. The lift $\pi^* E$ is a trivial bundle, $\pi^* E \simeq \widetilde{M} \times \mathbb{R}^s$, where the trivialization is uniquely determined up to an automorphism of $\mathbb{R}^s$. The natural projection $\hat{\pi} : \pi^* E \longrightarrow E$ is given in coordinates by the formula*

$$U \times \mathcal{G} \times \mathbb{R}^s \ni (z, g, y) \longmapsto (z, gy) \in U \times \mathbb{R}^s.$$

The space $L^2(E)$ (where the measure on E is locally chosen as the direct product of the measure on M constructed in the preceding subsection by the standard Lebesgue measure in the fibers) can be identified with the subspace $L^2_{\mathcal{G}}(\pi^* E) \subset L^2(\pi^* E)$ formed by $\mathcal{G}$-invariant functions $u(x, y)$, i.e., functions satisfying the condition

$$u(\sigma x, \sigma y) = u(x, y), \qquad x \in \widetilde{M}, \quad y \in \mathbb{R}^s, \quad \sigma \in \mathcal{G}.$$

Definition 2.2. A bounded operator

$$A : L^2(E) \longrightarrow L^2(E)$$

is said to be *translation invariant* if it is the restriction to $L^2_{\mathcal{G}}(\pi^* E) \simeq L^2(E)$ of a bounded $\mathcal{G}$-invariant operator

$$\widetilde{A} : L^2(\pi^* E) \longrightarrow L^2(\pi^* E) \tag{2.3}$$

that is invariant under translations in $\mathbb{R}^s$:

$$[\widetilde{A}u](x, y+t) = \widetilde{A}[u(x, y+t)], \quad \forall t \in \mathbb{R}^s.$$

Proposition 2.3. *If $A : L^2(E) \longrightarrow L^2(E)$ is a translation-invariant operator, then the corresponding operator* (2.3) *is unique.*

Proof. The proof will be given in the appendix. □

Remark 2.4. Since the trivialization of $\pi^* E$ is uniquely determined up to an automorphism of $\mathbb{R}^s$ (independent of the point of the base $\widetilde{M}$), the notion of a translation-invariant operator in $\widetilde{E}$ is well defined (automorphisms of $\mathbb{R}^s$ take translations to translations). It is here where the requirement of minimality of the structure group is important: without it, there would be several trivializations of $\pi^* E$ not taken to each other by a constant automorphism of the fiber, and the notion of translation-invariant operator would be ambiguous.

The translation-invariant operator (2.3) can be represented in the form

$$\widetilde{A} = B\left(-i\frac{\partial}{\partial y}\right), \tag{2.4}$$

where

$$B(q) : L^2(\widetilde{M}) \longrightarrow L^2(\widetilde{M}), \quad q \in \mathbb{R}^s, \tag{2.5}$$

is a bounded operator-valued function that is (at least) strongly measurable in q [13, Proposition 16].

Definition 2.5. The function (2.5) is called the *symbol* of the translation-invariant operator A and is denoted by $\sigma(A)$.

2.3. General local operators and localization principle

Local operators with parameters. Let X be a separable locally compact metric space equipped with a nonatomic Borel measure μ such that $\mu(U) > 0$ for any nonempty open set $U \subset X$. We deal with *local operators with a parameter* $q \in \mathbb{R}^s$ in the $C(X)$-module $H = L^2(X, d\mu)$. They are defined as operator families $A \in C(\mathbb{R}^s, \mathcal{B}H)$ such that for each $\varphi \in C_0(X)$ the commutator $[A(q), \varphi]$ belongs to the ideal $\mathcal{J} = C_0(\mathbb{R}^s, \mathcal{K}H)$ of compact-valued families decaying in norm as $q \to \infty$. Such families A obviously form a C^*-subalgebra in $C(\mathbb{R}^s, \mathcal{B}H)$, which will be denoted by $\mathcal{A} = \mathcal{A}(\mathbb{R}^s, \mathcal{B}H)$.

Localization principle. For $x \in X$, let $\mathcal{J}_x \subset \mathcal{A}$ be the ideal in $\mathcal{A}$ generated by the maximal ideal $\mathcal{I}_x \subset C(X)$ of functions vanishing at x, and let $p_x : \mathcal{A} \longrightarrow \mathcal{A}_x$ be the natural projection into the *local algebra* $\mathcal{A}_x = \mathcal{A}/\mathcal{J}_x$.

Theorem 2.6 (localization principle; cf. [13, Theorem 3])**.** *Suppose that the space X is compact. Then $\mathcal{J} = \bigcap_{x\in X} \mathcal{J}_x$, and hence an operator $A \in \mathcal{A}$ is*

1. *Compact with parameter q ($A \in \mathcal{J}$) if and only if all its local representatives $p_x(A) \in \mathcal{A}_x$ are zero.*
2. *Fredholm with parameter q (invertible modulo $\mathcal{J}$) if and only if all its local representatives $p_x(A) \in \mathcal{A}_x$ are invertible.*

The ideals $\mathcal{J}_x$ can be described as follows. For $U \subset X$ and $A \in \mathcal{A}$, set

$$\|A\|_U = \sup_{q\in\mathbb{R}^s} \left\| A(q) \,|_{H_U} \colon H_U \longrightarrow H \right\|, \quad \text{where } H_U = \{v \in H \colon \operatorname{supp} v \subset \overline{U}\}. \tag{2.6}$$

Proposition 2.7 (cf. [13, Proposition 4]). *The ideal $\mathcal{J}_x$ is the set of elements $A \in \mathcal{A}$ such that*

$$\lim_{U \downarrow x} \|A\|_U = 0. \tag{2.7}$$

(*Here the limit is taken over the filter of neighborhoods of x, i.e., over a sequence of open sets U shrinking to x.*)

Remark 2.8. Condition (2.7) is stated in [13] in the different form $\lim \|A\varphi\| = 0$, where $|\varphi| \leq 1$ and the support of φ shrinks to x; the two forms are easily seen to be equivalent. The assumption that X is compact is also easily removed.

Local representatives. Let us describe the range of the family $\{p_x\}_{x \in X}$ of "localizing homomorphisms." Consider a family $\{a_x \in \mathcal{A}_x\}_{x \in X}$. For each x, we arbitrarily pick up some representative $A_x \in a_x$. Proposition 2.7 has an immediate corollary:

Corollary 2.9. *The family $\{a_x\}$ has the form $a_x = p_x(A)$ for some $A \in \mathcal{A}$ if and only if for any $\varepsilon > 0$ each point $x \in X$ has a neighborhood $U(\varepsilon, x)$ such that*

$$\|A_x - A\|_{U(\varepsilon,x)} \leq \varepsilon. \tag{2.8}$$

This is not especially useful, because one has to know A in advance. Fortunately, one can give a criterion that does not resort to A.

Definition 2.10. The family $\{a_x\}$ is said to be *continuous* if for all $\varepsilon > 0$ and $x \in X$ there exist neighborhoods $U(\varepsilon, x)$ such that

$$\|A_y - A_{y'}\|_{U(\varepsilon,y) \cap U(\varepsilon,y')} \leq \varepsilon \quad \text{for any } y, y' \in X. \tag{2.9}$$

One can readily see that the definition of continuity is independent of the choice of $A_x \in a_x$ (but the neighborhoods $U(\varepsilon, x)$ depend on this choice).

Proposition 2.11 (cf. [13, Proposition 7]). (i) *The family $\{a_x\}$ is continuous if* (*and, in the case of compact X, only if*) *it has the form $a_x = p_x(A)$ for some $A \in \mathcal{A}$.*

(ii) *Under the assumptions of* (i), *if $a_x \in \mathcal{B}/\mathcal{J}$ for all $x \in X$, where $\mathcal{B} \subset \mathcal{A}$ is a C^*-subalgebra containing $\mathcal{J}$, then $A \in \mathcal{B}$.*

Remark. For the general localization principle, the topology on the disjoint union $\bigsqcup_x \mathcal{A}_x$ in which the families $\{p_x(A)\}_{x \in X}$, $A \in \mathcal{A}$, are exactly continuous sections of the projection $\bigsqcup_x \mathcal{A}_x \longrightarrow X$ is described e.g., in [2, 16]. In our special case, these sections admit the simpler description given above.

Infinitesimal operators. The study of local representatives of an operator $A \in \mathcal{A}$ is also local in the following sense. The class $p_{x_0}(A) \in \mathcal{A}_{x_0}$ remains unchanged if we multiply A (on the left or on the right) by any cutoff function $f \in C_0(X)$ such that $f(x_0) = 1$. (This can readily be derived from the fact that if $K \in \mathcal{J}$, then $\|K\|_U \to 0$ as $U \downarrow x$.) It follows that only what happens in an arbitrarily small neighborhood of x_0 is actually important. Consequently, if $\widetilde{X}$ is another metric space equipped with a measure $\widetilde{\mu}$ and a homeomorphism $f\colon U \longrightarrow \widetilde{U}$ of some neighborhood $U \subset X$ of x_0 onto a neighborhood $\widetilde{U} \subset \widetilde{X}$ of the point $\widetilde{x}_0 = f(x_0)$ is given such that f respects the classes of the measures μ and $\widetilde{\mu}$, then f^* induces an isomorphism between $\mathcal{A}_{x_0}$ and $\widetilde{\mathcal{A}}_{\widetilde{x}_0}$ and one can speak of local representatives of A in the algebra $\widetilde{A}$ of local operators with a parameter on $\widetilde{X}$.

We systematically use this construction in what follows; the space $\widetilde{X}$ will only reflect local properties of X near x_0 and is usually noncompact. Such local representatives, uniquely determined by certain additional conditions, will also be called *infinitesimal operators* to emphasize the fact that $X \neq \widetilde{X}$.

Example 2.12. If A is a pseudodifferential operator on a smooth manifold X, then one can identify a small neighborhood of x_0 with a small neighborhood of zero in $\widetilde{X} = T_{x_0}X$ via the geodesic exponential mapping and take the operator $\sigma(A)(x_0, -i\partial/\partial y)$ with constant coefficients on $T_{x_0}X$ for a local representative (infinitesimal operator) of A at x_0. (Here $\sigma(A)$ is the principal symbol of A and $y \in T_{x_0}X$.) This infinitesimal operator is uniquely determined by the condition of invariance with respect to the dilations $y \longmapsto \lambda y$ in $T_{x_0}X$.

2.4. **Definition and Properties of ΨDO.** Now we are in a position to define pseudodifferential operators with a parameter $q \in \mathbb{R}^s$ on a manifold M with corners. They will be local operators with a parameter in the sense of Sec. 2.3 possessing a number of additional properties.

Parameter dependence of ΨDO. First of all, we impose more restrictive conditions on the dependence of operators on the parameter $q \in \mathbb{R}^s$.

We treat $L^2(M)$ as a module over $C(M^\#)$ (by interpreting elements $u \in L^2(M)$ as functions on $M^\circ = M^{\#\circ}$) and consider the algebra $\mathcal{A}(\mathbb{R}^s, \mathcal{B}L^2(M))$ of operators with parameter $q \in \mathbb{R}^s$ local with respect to the action of $C(M^\#)$.

Definition 2.13. The *subalgebra* $\mathcal{A}_{scv} \equiv \mathcal{A}_{scv}(M, \mathbb{R}^s) \subset \mathcal{A}(\mathbb{R}^s, \mathcal{B}L^2(M))$ *of functions of slow compact variation* consists of operator families $A(q)$, $q \in \mathbb{R}^s$, satisfying the following conditions:

(1) The function $A(q)$ is of *compact variation*; that is, $A(q) - A(q') \in \mathcal{K}H$ for any $q, q' \in \mathbb{R}^s$.
(2) The function $A(q)$ *slowly varies at infinity* in the sense that for any $d > 0$ and $\varepsilon > 0$ there exists an $R > 0$ such that

$$\|A(y) - A(y')\| \leq \varepsilon \quad \text{whenever } |y - y'| < d \text{ and } |y| > R.$$

Proposition 2.14. *The set $\mathcal{A}_{scv}$ is a C^*-algebra, and every element $A(y) \in \mathcal{B}$ can be approximated by C^∞ functions of compact variation all of whose derivatives decay at infinity.*

Proof. The proof will be given in the appendix. □

Interior symbol. Let $A \in \mathcal{A}_{scv}(M, \mathbb{R}^s)$.

Definition 2.15. Let $x \in M^\circ$ be an interior point of M. We say that A is *Agranovich–Vishik at x* if, under the identification of a neighborhood of x in M° with a neighborhood of the origin in T_xM via a coordinate system near x_0, A has a local representative of the form

$$A_{x_0} = B\left(q, -i\frac{\partial}{\partial y}\right), \quad y \in T_{x_0}M,$$

where $B(q, \xi)$ is a function continuous for $|q| + |\xi| \neq 0$ and zero-order homogeneous:

$$B(\lambda q, \lambda \xi) = B(q, \xi), \quad \lambda \in \mathbb{R}_+.$$

The function $B(q, \xi)$ is called the *interior symbol* of A and is denoted by

$$\sigma_0(A)(x, \xi, q) := B(q, \xi).$$

Essentially, the definition says that at the point x the operator A is a parameter-dependent pseudodifferential operator with continuous symbol.

Proposition 2.16. *If A is Agranovich–Vishik at x, then $\sigma_0(A)$ is a well-defined function on $T_x^*M \times \mathbb{R}^s$ outside zero (i.e., its existence and form is independent of the choice of the coordinate system).*

Sketch of proof. The operator

$$\widehat{B} = B\left(q, -i\frac{\partial}{\partial y}\right)$$

behaves as desired under linear changes of the variable y. Thus, essentially, one should prove that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism with identity differential at the origin, then $\widehat{B}$ and $(f^*)^{-1}\widehat{B}f^*$ define the same element in the local algebra $\mathcal{A}_0$. To prove this, we approximate B by smooth classical symbols and use the theorem on the change of variables in a classical pseudodifferential operator. □

Face symbols. From now on, we choose and fix a compatible system of exponential maps from the normal bundles of the faces into their neighborhoods in M. Our definition of face symbols and of ψDO tacitly depends on the choice of this system. The questions concerning the invariance of the definition will be discussed elsewhere.

Let again $A \in \mathcal{A}_{scv}(M, \mathbb{R}^s)$, and let $z \in F^{\#}$ be a point of open face $F^{\#}$ dual to a face F of positive codimension $d \geq 1$ in M. Some neighborhood U of F can be identified via the exponential map with a neighborhood of the zero section in N_+F or (additionally applying the logarithmic map) with a neighborhood of the point at infinity in the positive quadrant inNF. Hence we have the embedding

$$L^2(M)|_U \subset L^2(N_+F) \simeq L^2(NF),$$

which implies that it suffices to localize the operator A at the point $z \in F^{\#}$ in the space $L^2(NF)$.

Definition 2.17. We say that the operator $A(q)$ has a *translation-invariant infinitesimal operator* at the point $z \in F^{\#}$ if in $L^2(NF)$ there exists a translation-invariant operator $A_\infty(q)$ (see Definition 2.2) belonging to the same coset in the local algebra $\mathcal{A}_z$. The symbol (Definition 2.5) of $A_\infty(q)$ will be called the *symbol of $A(q)$ at z* and will be denoted by $\sigma_z(A)$.

Theorem 2.18. *If $A(q)$ has a translation-invariant infinitesimal operator, then it is unique. Thus the symbol $\sigma_z(A)$ is well defined. It is a $\mathfrak{S}_F$-invariant operator-valued function on $\mathbb{R}^d \times \mathbb{R}^s$ with values in $\mathcal{B}L^2(\widetilde{F})$, where $\widetilde{F}$ is the principal $\mathfrak{S}_F$-covering over F trivializing NF.*

Proof will be given in the Appendix. □

Pseudodifferential operators. Let M be a manifold with corners.

Definition 2.19. The space $\Psi(M) \equiv \Psi(M, \mathbb{R}^s)$ of pseudodifferential operators consists of operator families $A(q)$ satisfying the following conditions:

(1) $A(q) \in \mathcal{A}_{scv}(M, \mathbb{R}^s)$.
(2) For each interior point $x \in M^\circ$, the family $A(q)$ is Agranovich–Vishik at x.
(3) For each face F of codimension $d = d(F) > 0$ in M, the family $A(q)$ has a $\mathfrak{S}_F$-invariant symbol $\sigma_z(A)$ in the sense of Definition 2.17 at each point $z \in F^{\#}$, and $\sigma_z(A)$ is independent of z. Moreover, $\sigma_z(A) \in \Psi(\widetilde{F}, \mathbb{R}^{d+s})$; i.e., the symbol $\sigma_z(A)$ is a $\mathfrak{S}_F$-invariant ψDO with parameters $(q,p) \in \mathbb{R}^s \times \mathbb{R}^d$ on the manifold $\widetilde{F}$ with corners, the covering of $\overline{F}$ trivializing the bundle $N\overline{F}$.

Since the symbol $\sigma_z(A)$ is independent of $z \in F^{\#}$, it will be denoted by $\sigma_F(A)$ in what follows. The interior symbol will be denoted by $\sigma_0(A)$; it is defined on the interior of $T^*M \times \mathbb{R}^s$ minus the zero section.

Main theorem of the calculus. The localization principle (Theorem 2.6) readily implies the following assertion.

Theorem 2.20 (main theorem of the calculus)**.** *A pseudodifferential operator A on a compact manifold M with corners is uniquely determined modulo the ideal $\mathcal{J}$ of compact operators with parameters by the symbol tuple $(\sigma_0(A), \{\sigma_F(A)\})$, where F runs over all faces of positive codimension, modulo compact operators. The map*

$$\sigma\colon A \longmapsto (\sigma_0(A), \{\sigma_F(A)\})$$

that takes each ψDO $A \in \Psi(M, \mathbb{R}^s)$ to it symbol tuple is a C^*-algebra homomorphism.

The symbol algebra. Now let us describe the symbol algebra, i.e., the range of the symbol map σ. In other words, we should indicate conditions on the interior symbol and the face symbols on faces of positive codimension necessary and sufficient for the existence of a ψDO with these symbols. To avoid awkward formulas, we first do so for the case in which the normal bundles of all faces are trivial and then indicate the modifications needed in the general case.

Thus let M be a manifold with corners such that the normal bundle NF is trivial for all faces F of M.

Let the following data be given:

- For each interior point $x \in M^\circ$, a continuous zero-order homogeneous function σ_x on $(T_x^*M \times \mathbb{R}^s) \setminus 0$.
- For each face F of codimension $d > 0$, a pseudodifferential operator $\sigma_F \in \Psi(F, \mathbb{R}^{d+s})$.

Theorem 2.21 (description of the symbol algebra)**.** *For the existence of a ψDO $A \in \Psi(M, \mathbb{R}^s)$ such that*

$$\sigma_0(A) = \sigma_x \quad \text{on } (T_x^*M \times \mathbb{R}^s) \setminus 0 \text{ for each } x \in M^\circ, \tag{2.10}$$

$$\sigma_F(A) = \sigma_F \quad \text{for each face } F \text{ of positive codimension,} \tag{2.11}$$

the following conditions are necessary and sufficient:

(1) *The functions σ_x form a continuous zero-order homogeneous function on the interior of $(T^*M \times \mathbb{R}^s) \setminus 0$ and extend by continuity to a continuous function (which we denote by σ_0) on the whole space $(T^*M \times \mathbb{R}^s) \setminus 0$.*

(2) *The restriction of σ_0 to the boundary satisfies the compatibility conditions*

$$\sigma_0 \big|_F = \sigma_0(\sigma_F) \quad \text{for each face } F \text{ of positive codimension,} \tag{2.12}$$

where the left-hand side is the restriction of σ_0 to $T^*M|_F \oplus \mathbb{R}^s$, naturally identified with $T^*F \oplus N^*F \oplus \mathbb{R}^s = T^*F \oplus \mathbb{R}^{d+s}$.

(3) *If $F_1 \succ F_2$ are two adjacent faces of M and Γ is a face of F_1 mapped into F_2 under the immersion of F_1 in M, then*

$$\sigma_\Gamma(\sigma_{F_1}) = \sigma_{F_2}. \tag{2.13}$$

Proof. First, note that routine computations based on composition formulas for pseudodifferential operators and standard norm estimates show that, being quantized, the symbols $\sigma_{x_0}(q,p)$ and $\sigma_F(q,\xi)$ give rise to the local representatives $\widehat{\sigma}_x = \sigma_{x_0}(q, -i\partial/\partial x)$ and $\widehat{\sigma}_F = \sigma_F(q, -i\partial/\partial t)$ that belong to $\mathcal{A}_{csv}$.

By Proposition 2.11, to prove the theorem it remains to establish that conditions (1)–(3) are exactly equivalent to the continuity of this family of local representatives in the sense of Definition 2.10.

(a) Let us show that the function σ_x continuously depends on x in the interior of M. Localizing our considerations, we can assume that $M = \mathbb{R}^n$. The family σ_x is continuous if and only if for each $\varepsilon > 0$ each point has a neighborhood

$U(\varepsilon, x)$ such that $\|\widehat{\sigma}_x - \widehat{\sigma}_y\|_{U(\varepsilon,x)\cap U(\varepsilon,y)} \le \varepsilon$ for any x and y. The intersection $U = U(\varepsilon, x) \cap U(\varepsilon, y)$ is necessarily nonempty if $y \in U(\varepsilon, x)$. Since the operator $\widehat{\sigma}_x - \widehat{\sigma}_y$ is dilatation invariant, it follows that

$$\|\widehat{\sigma}_x - \widehat{\sigma}_y\|_U = \|\widehat{\sigma}_x - \widehat{\sigma}_y\| = \max_p |\sigma_x - \sigma_y|$$

(provided that U is nonempty). Combining this with the homogeneity of σ_x in (p, q), we see that the continuity of the family of local representatives in the sense of Definition 2.10 is equivalent to the continuity of the interior symbol. This is of course well known from the theory of ψDO on smooth manifolds.

(b) Let us show that the interior symbol is continuous up to the boundary and satisfies the compatibility conditions (2.12) there. Fix a point $z_0 \in F$. Multiplying by a cutoff function $f \in C(M)$, we can study the problem assuming that $F = \mathbb{R}^{n-d}$ and $M^\circ = \mathbb{R}^{n-d} \times \mathbb{R}^d$ (here we use the logarithmic coordinates $y \in \mathbb{R}^d$, see (1.4) on the fibers of the normal bundle of F). Let $x_0 \in F$. Applying Corollary 2.9 and using the Fourier transform with respect to $\mathbb{R}^d$, we see that for each $\varepsilon > 0$ there is a neighborhood U_ε of x_0 in F such that

$$\|\widehat{\sigma}_F - \widehat{\sigma}_{x_0}(\sigma_F)\|_{U_\varepsilon \times \mathbb{R}^d} < \varepsilon. \tag{2.14}$$

(Here $\sigma_{x_0}(\sigma_F)$ is the symbol $\sigma_0(\sigma_F)$ restricted to the fiber over x_0.)

On the other hand, the continuity of the family of local representatives on M near $F^\#$ is equivalent to the existence of a neighborhood W_ε of the point at infinity on the diagonal of the positive quadrant in $\mathbb{R}^d$ such that

$$\|\widehat{\sigma}_F - \widehat{\sigma}_x\|_{(F\times W_\varepsilon)\cap U(\varepsilon,x)} < \varepsilon. \tag{2.15}$$

For $x \in U_\varepsilon \times W_e$, using the triangle inequality, from (2.14) and (2.15) we conclude that

$$\|\widehat{\sigma}_x - \widehat{\sigma}_{x_0}(\sigma_F)\|_U < 2\varepsilon \tag{2.16}$$

on the nonempty set $U = (U_\varepsilon \times W_e) \cap U(\varepsilon, x)$. Arguing as above, we see that $|\sigma_x - \sigma_{x_0}(\sigma_F)| \le 2\varepsilon$ for these x.

(c) In a similar way, one shows that condition (3) also follows from the continuity of local representatives and finally concludes that conditions (1)–(3) together are equivalent to the continuity. We leave the details to the reader. $\square$

Remark 2.22. In particular, it follows from the compatibility condition that the symbol on a face of positive codimension determines the symbols on all adjacent faces of larger codimension.

Let us now discuss how the compatibility conditions should be modified if the normal bundles of the faces are not trivial.

Let again $\overline{F}_1 \succ \overline{F}_2$ be two adjacent faces of M, and let Γ be a face of $\overline{F}_1$ covering $\overline{F}_2$ (there can be several such faces). The symbols $\sigma_{F_1}(A)$ and $\sigma_{F_2}(A)$ of A are operators with parameters on the minimum coverings $\widetilde{F}_1$ and $\widetilde{F}_2$ trivializing the bundles NF_1 and NF_2, respectively. Let $\widetilde{\Gamma}$ be the lift of Γ to $\widetilde{F}_1$. The symbol $\sigma_{\widetilde{\Gamma}}(\sigma_{F_1}(A))$ is defined on the covering $\widetilde{\widetilde{\Gamma}}$ trivializing the bundle $N\widetilde{\Gamma}$. The composite covering $\widetilde{\widetilde{\Gamma}} \longrightarrow F_2$ trivializes NF_2 (since it trivializes both direct summands $N\widetilde{\Gamma}$ and $NF_1|_\Gamma$). Since the trivializing covering $\widetilde{F}_2 \longrightarrow F_2$ is minimal and hence universal, there exists a unique (modulo permutation of the sheets) covering $\widetilde{\widetilde{\Gamma}} - \to \widetilde{F}_2$ making

the triangle

(2.17)
$$\begin{tikzcd} \widetilde{\widetilde{\Gamma}} \arrow[r, dashed, "\pi"] \arrow[dr] & \widetilde{F}_2 \arrow[d] \\ & F_2 \end{tikzcd}$$

commute.

Let $L^2_{inv}(\pi)$ be the subspace of $L^2(\widetilde{\Gamma})$ consisting of functions invariant with respect to permutations of the sheets of π. The compatibility condition (2.13) in this situation is generalized to

$$\sigma_{\widetilde{\Gamma}}(\sigma_{F_1}(A))|_{L^2_{inv}(\pi)} = \sigma_{F_2}(A). \tag{2.18}$$

The counterpart of the compatibility condition (2.12) reads

$$\sigma_0(\sigma_F(A)) = \pi_F^* \left[\sigma_0(A)|_{T^*F}\right], \tag{2.19}$$

where $\pi_F : T^*\widetilde{F} \longrightarrow T^*F$ is the covering associated with the covering $\widetilde{F} \longrightarrow F$.

Appendix A. Proofs of Some Assertions

Proof of Proposition 1.2. For brevity, we write

$$F = \Gamma_j^\circ(M) \quad d = d_j.$$

Let $U \simeq \overline{\mathbb{R}}^s_+ \times \mathbb{R}^{n-s}$ be a coordinate neighborhood on M. If the intersection $U \cap F$ is nonempty (which can happen only for $s \geq d$), then it consists of finitely many ($\leq C_s^d$) connected components of the form $V \simeq \mathbb{R}^{s-d}_+ \times \mathbb{R}^{n-s}$, where the open coordinate quadrant $\mathbb{R}^{s-d}_+$ of dimension $s-d$ is singled out in $\overline{\mathbb{R}}^s_+$ by the relations

$$x_{j_1} = \cdots = x_{j_d} = 0, \quad x_{j_{d+1}}, \dots, x_{j_s} > 0$$

for some (depending on V) permutation $j_1, \dots, j_s$ of the indices $1, \dots, s$. If we accordingly permute the standard coordinates $x_1, \dots x_n$ in U, setting

$$\rho_1 = x_{j_1}, \dots, \rho_d = x_{j_d},$$
$$y_{d+1} = x_{j_{d+1}}, \dots, y_s = x_{j_s}, y_{s+1} = x_{s+1}, \dots, y_n = x_n,$$

then the variables $y = (y_{d+1}, \dots, y_n)$ are coordinates in V and the variables $\rho = (\rho_1, \dots, \rho_d)$ are defining functions of V for the embedding $V \subset U$ (and local defining functions of $\overline{F}$); i.e., locally the face is given by the conditions $\rho = 0$.

We take a finite cover of M by coordinate neighborhoods U and various connected components $V \subset U \cap F$ and obtain a finite atlas

$$\left\{\left(V,\, y : V \longrightarrow \mathbb{R}^{s-d}_+ \times \mathbb{R}^{n-s}\right)\right\}$$

on $\overline{F}$ such that associated with each coordinate neighborhood V of this atlas is a pair (U, V) and coordinates (ρ, y) in U. Let $\widetilde{V}$ be another coordinate neighborhood $(\widetilde{U}, \widetilde{V})$ with coordinates $(\widetilde{\rho}, \widetilde{y})$ in $\widetilde{U}$, and suppose that the intersection $V \cap \widetilde{V}$ is nonempty. The change of variables

$$\widetilde{y} \circ y^{-1} : y(V \cap \widetilde{V}) \longrightarrow \widetilde{y}(V \cap \widetilde{V})$$

is obtained by restriction to $y(V \cap \widetilde{V})$ from the change of coordinates $(\rho, y) \longmapsto (\widetilde{\rho}, \widetilde{y})$ on the intersection of the coordinate neighborhoods U and $\widetilde{U}$ on M and hence has a smooth continuation to the closure of the set $y(V \cap \widetilde{V})$ in $\overline{\mathbb{R}}^{s-d}_+ \times \mathbb{R}^{n-s}$. (The continuation is obtained by restriction of the same change of coordinates to the closure.) These continuations determine the transition functions of some compact manifold $\overline{F}$ with corners whose local models are $\overline{\mathbb{R}}^{s-d}_+ \times \mathbb{R}^{n-s}$ and into which F is naturally embedded as a dense open submanifold. This manifold $\overline{F} =: \Gamma_j(M)$

is the *closed face* of M corresponding to the open face $\Gamma_j^\circ(M)$. The embedding $\Gamma_j^\circ(M) \subset M$ extends by continuity to $\Gamma_j(M)$; the resulting mapping is in general an immersion with self-intersections.

Proof of Proposition 1.4. It suffices to write out a natural invariant pairing; this can be done in the coordinates (ρ, y): for a form

$$\omega = \sum a_j \rho_j^{-1} d\rho_j \in N^*\overline{F}$$

and a vector

$$\xi = (b_1, \dots, b_d) \in N\overline{F},$$

we set

$$\langle \omega, \xi \rangle = \sum a_j b_j.$$

Under changes of coordinates, the components of ξ and ω are subjected to the same permutation, and the defining functions ρ_j are multiplied by nonzero numbers (the diagonal entries of the matrix $\Lambda(y)$), which does not affect the logarithmic derivatives, so that the numbers a_j remain the same. Thus the pairing is independent of the choice of coordinates.

Proof of Theorem 1.5. We need the following simple lemma.

Lemma A.1. *If smooth mappings*

$$g_j : \mathbb{R}_+^k \longrightarrow \mathbb{R}_+^k, \quad g_j(0) = 0,\ j = 1, \dots, l,$$

are diffeomorphisms in a neighborhood of the origin, all matrices $g_j'(0)\big(g_i'(0)\big)^{-1}$ *are diagonal, and* $\lambda_1, \dots, \lambda_l$ *are nonnegative numbers at least one of which is nonzero, then the mapping*

$$g \equiv \sum_{j=1}^{l} \lambda_j g_j : \mathbb{R}_+^k \longrightarrow \mathbb{R}_+^k$$

is also a diffeomorphism in a neighborhood of the origin.

Indeed, the only nontrivial assertion is that g is epimorphic, but this can be verified as follows. Since the matrices $g_j'(0)\big(g_i'(0)\big)^{-1}$ are diagonal, it follows that all g_j take any given coordinate quadrant of arbitrary dimension to one and the same coordinate quadrant.

The lemma suggests that one can construct the desired mapping $f = f_j$ specifying it locally by the formula

$$\rho = r, \tag{A.1}$$

where ρ is a local tuple of defining functions of the face $\Gamma_j(M)$ and r are the corresponding coordinates in the fiber of $N_+\Gamma_j(M)$ and then gluing the local mappings with the use of a partition of unity.

We implement this idea and construct the mapping f, successively extending the set on which it is defined. Suppose that f has already been defined over some set $O \subset \Gamma_j(M)$, and let V be a local chart on $\Gamma_j(M)$ with the corresponding pair $(U = V \times \overline{\mathbb{R}}_+^{d_j}, V)$, so that over V the mapping can be given by formula (A.1). Let (φ_O, φ_V) be a nonnegative partition of unity on $O \cup V$ subject to the cover by O and V. we construct the map over $O \cup V$ by setting

$$f_{O\cup V} = \begin{cases} f_O & \text{over } O \setminus \operatorname{supp} \varphi_V, \\ \varphi_O f_O + \varphi_V f_V & \text{over } V, \end{cases}$$

where the addition in the second line is carried out in the fibers of $U \longrightarrow V$ (and is well defined in a sufficiently small neighborhood of V). Since $\varphi_O = 1$ on $V \setminus \operatorname{supp} \varphi_V$, it follows that both definitions are compatible on the set where they

apply simultaneously, and the lemma now implies that we have defined a mapping with the desired properties over $O \cup V$.

To complete the proof of Theorem 1.5, it suffices to start from an empty set O and successively add to it all charts from a finite atlas on $\Gamma_j(M)$. To obtain compatible (in the sense that diagram (1.3) commutes) exponential mappings for all faces, one should start from faces of maximal codimension.

Proof of Proposition 2.3. Let us proceed from the operator $\widetilde{A}$ to the symbol $B(p) = \sigma(A)$. We have to prove its uniqueness; it is a consequence of the following assertion, which we state in general form.

Lemma A.2. *Let a finite group G act on the space $L^2(\mathbb{R}^s; H)$, where H is a Hilbert space, by the formula*

$$[T_g f](p) = S_g f(\sigma_g^{-1} p), \quad p \in \mathbb{R}^s,$$

*where S is a representation of G on H and σ is a **faithful** representation of the same group on $\mathbb{R}^s_p$. Let*

$$A(p)\colon H \longrightarrow H, \quad p \in \mathbb{R}^s,$$

be a continuous operator-valued function. Then the following assertions hold.

(ii) *If $A(p)f(p) = 0$ for almost all p for any element $f \in L^2(\mathbb{R}^s; H)$ such that*

$$T_g f = f \quad \forall g \in G, \tag{A.2}$$

then $A(p) = 0$ for all p.

(ii) *If the operator $A : L^2(\mathbb{R}^s; H) \longrightarrow L^2(\mathbb{R}^s; H)$ induced by the multiplication by $A(p)$ preserves the subspace of invariant functions* (A.2), *then it is G-invariant, i.e., satisfies*

$$A(p) = S_g^{-1} A(\sigma_g(p)) S_g, \qquad \forall g \in G.$$

Proof. 1. Since $A(p)$ is continuous, it suffices to prove the desired relation on a dense set of values of p. For this set we take $\Omega = \mathbb{R}^s \setminus \left(\bigcup_j \operatorname{fix} \sigma_g\right)$, where $\operatorname{fix} \sigma_g$ is the set of fixed points of σ_g (which is at most a hyperplane, since σ is faithful). Let $p_0 \in \Omega$ and $v \in H$. We claim that $A(p_0)v = 0$. Indeed, let U be a sufficiently small neighborhood of p_0 such that $U \subset \Omega$ and hence

$$\sigma_g(U) \cap \sigma_h(U) = \varnothing \quad \text{for} \quad g \neq h, \quad g, h \in G.$$

Then the function

$$f(p) = \begin{cases} S_g v & \text{if } p \in \sigma_g(U) \quad \text{for some } g \in G, \\ 0 & \text{otherwise} \end{cases}$$

is well defined. It is G-invariant, so that $A(p)f(p) = 0$ for almost all p and hence $A(p_0)v = 0$ (since $A(p)f(p) \equiv A(p)v$ is continuous in U).

2. The second assertion of the lemma is proved by the same method. Namely, it follows from the G-invariance of $f(p)$ and the assumptions of the lemma that $T_h(A(p)f(p)) = A(p)f(p)$ for each $h \in G$. In turn, this implies $T_h A(p) T_h^{-1} f(p) = A(p)f(p)$, whence for $p = p_0$ we obtain the desired relation

$$S_h A(\sigma_{h^{-1}}(p_0)) S_h^{-1} v = A(p_0)v.$$

□

Remark A.3. It is important that the representation σ is faithful. Without this assumption, the lemma fails. (One can only prove that $A(p)v = 0$ for elements v invariant under S_g for $g \in \ker \sigma$). In terms of our problem, this means that one should always reduce the structure group of th bundle $N_+\overline{F}$ to the *minimum possible* subgroup.

Proof of Proposition 2.14. The first assertion is trivial. The proof of the second assertion goes by the following scheme:

(1) The approximation is defined in the standard way as the convolution with a smooth function with unit integral and small support.
(2) All derivatives of an approximating function are bounded.
(3) Since the original family slowly varies at infinity, it follows that the first derivative of the approximating family decays at infinity.
(4) In turn, (2) and (3) imply that all derivatives decay at infinity. □

Proof of Theorem 2.18. It suffices to prove uniqueness for the case in which $A(q) \equiv 0$. Consider a sequence $\varphi_n \in C_0(N\overline{F})$ strongly convergent to the identity operator. Then $\varphi_n A_\infty \varphi_n$ strongly converges to A_∞. Passing to the cover $N\widetilde{F}$, we see that the product $\widetilde{\varphi}_n \widetilde{A}_\infty \widetilde{\varphi}_n$ of the corresponding lifted operators strongly converges to $\widetilde{A}_\infty$. On the other hand, for a given n, let $a_j \in \mathbb{R}^d$ be a sequence of vectors such that the supports of the functions $t^*_{a_j}\widetilde{\varphi}_n$, where t_{a_j} is the shift by the vector $a_j \in \mathbb{R}^d$, lie as $j \to \infty$ in an arbitrarily small neighborhood of some of the preimages z_* of the point z (i.e., go to infinity in the positive quadrant along the corresponding ray). These functions are no longer $\mathfrak{S}_F$-invariant. However, one can show that there exist functions ψ_j bounded by 1 with supports shrinking to z such that their invariant lifts satisfy the condition

$$\widetilde{\psi}_j t^*_{a_j}\widetilde{\varphi}_n = t^*_{a_j}\widetilde{\varphi}_n.$$

Then, according to the properties of the local algebra $\mathcal{A}_z$, we have

$$\begin{aligned}(t^*_{a_j}\widetilde{\varphi}_n)\widetilde{A}_\infty(t^*_{a_j}\widetilde{\varphi}_n) &= (t^*_{a_j}\widetilde{\varphi}_n)\widetilde{\psi}_j\widetilde{A}_\infty\widetilde{\psi}_j(t^*_{a_j}\widetilde{\varphi}_n)\\ &= (t^*_{a_j}\widetilde{\varphi}_n)\widetilde{\psi_j A_\infty \psi_j}(t^*_{a_j}\widetilde{\varphi}_n) \to 0\end{aligned}$$

as $j \to \infty$ (convergence in norm). Indeed, the extreme factors are uniformly bounded, and the middle factor converges to zero, since A_∞ represents the zero class. Thus

$$(t^*_{a_j}\widetilde{\varphi}_n)\widetilde{A}_\infty(t^*_{a_j}\widetilde{\varphi}_n) = t^*_{a_j} \circ \widetilde{\varphi}_n\widetilde{A}_\infty\widetilde{\varphi}_n \circ t^{*-1}_{a_j} \to 0.$$

(We have used the translation invariance of A.) We see that $\widetilde{\varphi}_n\widetilde{A}_\infty\widetilde{\varphi}_n = 0$ and, passing to the limit as $n \to \infty$, find that $\widetilde{A}_\infty$ and hence A_∞ are zero.

References

[1] U. Bunke, *Index theory, eta forms, and Deligne cohomology*, arXiv: math.DG/0201112.

[2] J. Dauns and K. H. Hofmann, *Representation of rings by sections*, Memoirs of the American Mathematical Society, No. 83, Amer. Math. Soc., Providence, R.I., 1968.

[3] T. Krainer, *Elliptic boundary problems on manifolds with polycylindrical ends*, arXiv: math.AP/0508516, 2005.

[4] R. Lauter and S. Moroianu, *The index of cusp operators on manifolds with corners*, Ann. Global Anal. Geom. **21** (2002), no. 1, 31–49.

[5] P.-Y. Le Gall and B. Monthubert, *K-theory of the indicial algebra of a manifold with corners*, *K*-Theory **23** (2001), no. 2, 105–113.

[6] P. Loya, *The index of b-pseudodifferential operators on manifolds with corners*, Ann. Global Anal. Geom. **27** (2005), no. 2, 101–133 (English).

[7] R. Melrose, *Analysis on manifolds with corners*, Lecture Notes, MIT, Cambrige, MA, 1988, Preprint.

[8] ———, *Pseudodifferential operators, corners, and singular limits*, Proceedings of the International Congress of Mathematicians, Kyoto (Berlin–Heidelberg–New York), Springer-Verlag, 1990, pp. 217–234.

[9] R. Melrose and V. Nistor, *K-theory of* $\mathbf{C}^*$*-algebras of b-pseudodifferential operators*, Geom. Funct. Anal. **8** (1998), no. 1, 88–122.

[10] R. Melrose and P. Piazza, *Analytic K-theory on manifolds with corners*, Adv. in Math. **92** (1992), no. 1, 1–26.

[11] B. Monthubert, *Groupoids and pseudodifferential calculus on manifolds with corners*, J. Funct. Anal. **199** (2003), no. 1, 243–286.

[12] B. Monthubert and V. Nistor, *A topological index theorem for manifolds with corners*, arXiv: math.KT/0507601, 2005.

[13] V. Nazaikinskii, A. Savin, and B. Sternin, *Pseudodifferential operators on stratified manifolds*, arXiv: math.AP/0512025, 2005.

[14] V. Nistor, *An index theorem for gauge-invariant families: The case of solvable groups*, Acta Math. Hungarica **99** (2003), no. 2, 155–183.

[15] B. A. Plamenevsky and V. N. Senichkin, *Representations of C^*-algebras of pseudodifferential operators on piecewise-smooth manifolds*, Algebra i Analiz **13** (2001), no. 6, 124–174.

[16] N. Vasilevski, *Local principles in operator theory*, Lineinye operatory v funktsionalnykh prostranstvakh. Tez. dokl. Severo-Kavkaz. reg. konf., Groznyi, 1989, pp. 32–33.

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